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Multidimensional extension of the generalized Chowla–Selberg formula

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Abstract

After recalling the precise existence conditions of the zeta function of a pseudodifferential operator, and the concept of reflection formula, an exponentially convergent expression for the analytic continuation of a multidimensional inhomogeneous Epstein-type zeta function of the general form

$$\zeta_{A,\vec{b},q}(s) = \sum_{\vec{n}\in \mathbf{Z}^p} (\vec{n}^T A \vec{n} + \vec{b}^T \vec{n} + q)^{-s},$$

with A the $p \times p$ matrix of a quadratic form, $\vec{b}$ a p vector and q a constant, is obtained. It is valid on the whole complex s-plane, is exponentially convergent and provides the residua at the poles explicitly. It reduces to the famous formula of Chowla and Selberg in the particular case $p = 2$, $\vec{b} = \vec{0}$, $q = 0$. Some variations of the formula and physical applications are considered.

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1. Introduction: Existence of the zeta function of a pseudodifferential operator (ΨDO)

A *pseudodifferential operator* A of order m on a manifold M_n is defined through its symbol $a(x,\xi)$, which is a function belonging to the space $S^m(\mathbf{R}^n \times \mathbf{R}^n)$ of $\mathbf{C}^\infty$ functions such that for any pair of multi-indexs α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}. \tag{1}$$

The definition of A is given, in the distribution sense, by

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi, \tag{2}$$

where f is a smooth function, $f \in \mathcal{S}$ [remember that $\mathcal{S} = \Big\{ f \in C^\infty(\mathbf{R}^n); \sup_x |x^\beta \partial^\alpha f(x)| < \infty,$ $\forall \alpha, \beta \in \mathbf{R}^n \}$], $\mathcal{S}'$ being the space of tempered distributions and $\hat{f}$ the Fourier transform of f. When $a(x,\xi)$ is a polynomial in ξ one gets a differential operator. In general, the order m can be complex. The *symbol* of a ΨDO has the form

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots, \tag{3}$$

being $a_k(x,\xi) = b_k(x)\, \xi^k$.

Pseudodifferential operators are useful tools, both in mathematics and in physics. They were crucial for the proof of the uniqueness of the Cauchy problem [1] and also for the proof of the Atiyah-Singer index formula [2]. In quantum field theory they appear in any analytical continuation process (as complex powers of differential operators, like the Laplacian) [3]. And they constitute nowadays the basic starting point of any rigorous formulation of quantum field theory through microlocalization, a concept that is considered to be the most important step towards the understanding of linear partial differential equations since the invention of distributions [4].

For A a positive-definite elliptic ΨDO of positive order $m \in \mathbf{R}$, acting on the space of smooth sections of an n-dimensional vector bundle E over a closed, n-dimensional manifold M, the *zeta function* is defined as

$$\zeta_A(s) = \mathrm{tr}\ A^{-s} = \sum_j \lambda_j^{-s}, \qquad \mathrm{Re}\ s > \frac{n}{m} \equiv s_0. \tag{4}$$

The quotient $s_0 = \dim M / \mathrm{ord}\, A$ is called the *abscissa of convergence* of $\zeta_A(s)$, which is proven to have a meromorphic continuation to the whole complex plane $\mathbf{C}$ (regular at s_0), provided that the principal symbol of A (that is $a_m(x,\xi)$) admits a *spectral cut*: $L_\theta =$

$\{\lambda \in \mathbf{C}; \operatorname{Arg}\lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\operatorname{Spec} A \cap L_\theta = \emptyset$ (the Agmon-Nierenberg condition). Strictly speaking, the definition of $\zeta_A(s)$ depends on the position of the cut L_θ, not so that of the determinant [5] $\det_\zeta A = \exp[-\zeta_A'(0)]$, which only depends on the homotopy class of the cut. The precise structure of the analytical continuation of the zeta function is known in general [6]. The only singularities it can have are *simple poles* at

$$s_k = (n-k)/m, \qquad k = 0, 1, 2, \ldots, n-1, n+1, \ldots. \tag{5}$$

The applications of the zeta-function definition of a determinant in physics are very important [7, 8]. A zeta function with the same meromorphic structure in the complex s-plane and extending the ordinary definition to operators of complex order $m \in \mathbf{C}\backslash\mathbf{Z}$ (it is clear that operators of complex order do not admit spectral cuts), has been obtained in Ref. [9]. The construction starts there from the definition of a trace, obtained as the integral over the manifold of the trace density of the difference between the Schwartz kernel of A and the Fourier transformed of a number of first homogeneous terms (in ξ) of the usual decomposition of the symbol (3) of A.

2. Exponentially convergent analytic continuation (with explicit poles and residua) of our zeta function

A fundamental property shared by zeta functions of any nature is the existence of a reflection formula. For a generic zeta function, $Z(s)$, it has the form:

$$Z(\omega - s) = F(\omega, s) Z(s), \tag{6}$$

and allows for its analytical continuation in a very easy way —what is the whole story of the zeta function regularization procedure (or at least the main part of it). But the analytically continued expression thus obtained is just another series, which has again a slow convergence behavior, of power series type (actually the same that the original series had, in its own domain of validity). Some years ago, the notorious mathematicians S. Chowla and A. Selberg found a formula, for the case $p = 2$, $\vec{b} = 0$ and $q = 0$ above [10], that yields *exponentially quick convergence, and not only in the reflected domain.* They were extremely proud of that formula —as one can appreciate by reading the original paper, where actually no hint about its derivation was given. In Ref. [11] we generalized this expression to inhomogeneous zeta functions, but staying always in *two* dimensions ($p = 2$), for this was commonly believed to be an unsurmountable restriction of the original formula (see, for instance, Ref. [12]). The generalization we carried out in [11] was already non trivial and a very detailed account of

this step is given in [13] (where a missprint of the original derivation is corrected). Finally, we have realized that an extension to an *arbitrary* number of dimensions is actually possible. In fact, here we shall obtain formulas for arbitrary p and arbitrary values of $\vec{b}$ and q.

The starting point will be *Poisson's resummation formula* in p dimensions, which arises from the distribution identity

$$\sum_{\vec{n}\in\mathbf{Z}^p} \delta(\vec{x}-\vec{n}) = \sum_{\vec{m}\in\mathbf{Z}^p} e^{i2\pi\vec{m}\cdot\vec{x}}. \tag{7}$$

(We shall indistinctly write $\vec{m}\cdot\vec{x} \equiv \vec{m}^T\vec{x}$ in what follows.) Applying this identity to the function

$$f(\vec{x}) = \exp\left(-\frac{1}{2}\vec{x}^T A\vec{x} + \vec{b}^T\vec{x}\right), \tag{8}$$

with A an invertible $p\times p$ matrix, and integrating then over $\vec{x}\in\mathbf{R}^p$, one obtains

$$\sum_{\vec{n}\in\mathbf{Z}^p} \exp\left(-\frac{1}{2}\vec{n}^T A\vec{n} + \vec{b}^T\vec{n}\right) = \frac{(2\pi)^{p/2}}{\sqrt{\det A}} \sum_{\vec{m}\in\mathbf{Z}^p} \exp\left[\frac{1}{2}\left(\vec{b}+2\pi i\vec{m}\right)^T A^{-1}\left(\vec{b}+2\pi i\vec{m}\right)\right]. \tag{9}$$

We are going to consider the following zeta function

$$\zeta_{A,\vec{c},q}(s) = {\sum_{\vec{n}\in\mathbf{Z}^p}}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A\left(\vec{n}+\vec{c}\right)+q\right]^{-s} \equiv {\sum_{\vec{n}\in\mathbf{Z}^p}}' \left[Q\left(\vec{n}+\vec{c}\right)+q\right]^{-s}, \quad \text{Re }(s) > \frac{p}{2}. \tag{10}$$

The aim is to obtain a formula that gives (the analytical continuation of) this multidimensional zeta function in terms of an exponentially convergent multiseries and which is valid in the whole complex plane, exhibiting the singularities (simple poles) of the meromorphic continuation —with the corresponding residua— explicitly. The prime in the summatories of Eq. (10) means that the point $\vec{n}=\vec{0}$ is to be excluded from the sum. Such restriction is irrelevant as long as $q\neq 0$ (the contribution of this single point being immediately obtainable), but to define the zeta function in this way is essential in order to be able to reach the limit $q\to 0$ (as we shall do later). The only condition on the matrix A is that it corresponds to a (non negative) quadratic form, that we call Q. The vector $\vec{c}$ is arbitrary, while q will (for the moment) be a positive constant.

2.1. The main expression (inhomogeneous case). Use of the Poisson resummation formula (9) yields, after some work, the following expression:

$$\begin{aligned}\zeta_{A,\vec{c},q}(s) \;=\;& \frac{(2\pi)^{p/2}q^{p/2-s}}{\sqrt{\det A}}\,\frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\,\Gamma(s)} \\ & \times {\sum_{\vec{m}\in\mathbf{Z}^p_{1/2}}}' \cos(2\pi\vec{m}\cdot\vec{c})\left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right),\end{aligned} \tag{11}$$

where K_ν is the modified Bessel function of the second kind and the subindex 1/2 in $\mathbf{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbf{Z}^p$ intervene in the sum. That is, if we take an $\vec{m} \in \mathbf{Z}^p$ we must then exclude $-\vec{m}$ (as simple criterion one can, for instance, select those vectors in $\mathbf{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive). Eq. (11) fulfills *all* the requirements demanded before. It is notorious to observe how the only pole of this inhomogeneous Epstein zeta function appears explicitly at $s = p/2$, where it belongs. Its residue is given by the formula:

$$\mathrm{Res}_{s=p/2}\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\sqrt{\det A}\ \Gamma(p/2)}. \tag{12}$$

With a bit of care, it is relatively simple to obtain the limit of expression (11) as $q \to 0$.

However, instead of proceeding in this way (what we shall do later), it is adviceable to construct first a direct recurrent formula for the case $q = 0$. This is certainly the natural option in such case, where no cut-off q exists to safeguard the t-integration (there is *no* way to use the Poisson formula on all p indices of $\vec{n}$ at once). However, we can still deal with this case by using the Poisson resummation formula on *some* of the p indices $\vec{n}$ only, say on just one of them, n_1. Poisson's formula on one index reduces to the celebrated Jacobi identity for the θ_3 function

$$\theta_3\left(z\,|\tau\right) = 1 + 2\sum_{n=1}^\infty q^{n^2}\cos(2nz), \qquad q = e^{\pi i\tau}, \quad |q| < 1, \quad \tau \in \mathbf{C}, \tag{13}$$

the identity being:

$$\theta_3\left(z\,|\tau\right) = \frac{1}{\sqrt{-i\tau}} e^{z^2/(\pi i\tau)}\theta_3\left(\frac{z}{\tau}\,\middle|\,\frac{-1}{\tau}\right), \tag{14}$$

or, in other words

$$\sum_{n=-\infty}^\infty e^{n^2\pi i\tau + 2niz} = \frac{1}{\sqrt{-i\tau}} \sum_{n=-\infty}^\infty e^{(z-n\pi)^2/(\pi i\tau)} \tag{15}$$

(for a classical reference see, e.g. Ref. [14]). Here z and τ are arbitrary complex, $z, \tau \in \mathbf{C}$, with the only restriction that $\mathrm{Im}\,\tau > 0$ (in order that $|q| < 1$). For the applications, it turns out to be better to recast the Jacobi identity as follows (with $\pi i\tau \to -t$ and $z \to \pi z$):

$$\sum_{n=-\infty}^\infty e^{-n^2t + 2\pi inz} = \sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^\infty e^{-\pi^2(n-z)^2/t}, \tag{16}$$

equivalently

$$\sum_{n=-\infty}^\infty e^{-(n+z)^2t} = \sqrt{\frac{\pi}{t}}\left[1 + \sum_{n=1}^\infty e^{-\pi^2n^2/t}\cos(2\pi nz)\right], \tag{17}$$

with $z, t \in \mathbf{C}$, $\mathrm{Re}\,t > 0$.

2.2. Recurrent expression (and the homogeneous case). Using this last formula on the first component, n_1, of the summation vector $\vec{n}$, we obtain (for the sake of simplicity of the final expressions, we shall just consider now the case $\vec{c}=\vec{0}$, but the result can be generalized to $\vec{c}\neq\vec{0}$ quite easily):

$$\zeta_{A,\vec{0},q}(s) = 2\sum_{n_1=1}^{\infty}\left(an_1^2+q\right)^{-s} + \frac{1}{\Gamma(s)}\sum_{\vec{n}_2\in\mathbf{Z}^{p-1}}{}' \left[\sqrt{\frac{\pi}{a}}\,\Gamma(s-1/2)\left(\vec{n}_2^T\Delta_{p-1}\vec{n}_2+q\right)^{1/2-s}\right. \tag{18}$$

$$\left. +\frac{4\pi^s}{a^{s/2+1/4}}\sum_{n_1=1}^{\infty}\cos\left(\frac{\pi n_1}{a}\vec{b}^T\vec{n}_2\right)n_1^{s-1/2}\left(\vec{n}_2^T\Delta_{p-1}\vec{n}_2+q\right)^{1/4-s/2}K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a}}\sqrt{\vec{n}_2^T\Delta_{p-1}\vec{n}_2+q}\right)\right],$$

which can be written as

$$\zeta_{A,\vec{0},q}(s) = \zeta_{a,\vec{0},q}(s) + \sqrt{\frac{\pi}{a}}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}\,\zeta_{\Delta_{p-1},\vec{0},q}(s-1/2) + \frac{4\pi^s}{a^{s/2+1/4}\,\Gamma(s)}\sum_{\vec{n}_2\in\mathbf{Z}^{p-1}}{}'$$

$$\sum_{n_1=1}^{\infty}\cos\left(\frac{\pi n_1}{a}\vec{b}^T\vec{n}_2\right)n_1^{s-1/2}\left(\vec{n}_2^T\Delta_{p-1}\vec{n}_2+q\right)^{1/4-s/2}K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a}}\sqrt{\vec{n}_2^T\Delta_{p-1}\vec{n}_2+q}\right). \tag{19}$$

This is clearly a recurrent formula in p, the number of dimensions, the first term of the recurrence being

$$\begin{aligned}\zeta_{a,\vec{0},q}(s) &= 2\sum_{n=1}^{\infty}\left(an^2+q\right)^{-s} = q^{-s}+\sqrt{\frac{\pi}{a}}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}\,q^{1/2-s}\\ &\quad +\frac{4\pi^s}{\Gamma(s)}a^{-1/4-s/2}q^{1/4-s/2}\sum_{n=1}^{\infty}n^{s-1/2}K_{s-1/2}\left(2\pi n\sqrt{\frac{q}{a}}\right).\end{aligned} \tag{20}$$

To take in these expressions the limit $q\to 0$ is immediate. One obtains:

$$\zeta_{A,\vec{0},0}(s) = 2a^{-s}\zeta(2s) + \sqrt{\frac{\pi}{a}}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}\,\zeta_{\Delta_{p-1},\vec{0},0}(s-1/2) + \frac{4\pi^s}{a^{s/2+1/4}\,\Gamma(s)}\sum_{\vec{n}_2\in\mathbf{Z}^{p-1}}{}'$$

$$\sum_{n_1=1}^{\infty}\cos\left(\frac{\pi n_1}{a}\vec{b}^T\vec{n}_2\right)n_1^{s-1/2}\left(\vec{n}_2^T\Delta_{p-1}\vec{n}_2\right)^{1/4-s/2}K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a}}\sqrt{\vec{n}_2^T\Delta_{p-1}\vec{n}_2}\right). \tag{21}$$

In the above formulas, A is a $p\times p$ symmetric matrix $A=\left(a_{ij}\right)_{i,j=1,2,\dots,p}=A^T$, A_{p-1} the $(p-1)\times(p-1)$ reduced matrix $A_{p-1}=\left(a_{ij}\right)_{i,j=2,\dots,p}$, a the component $a=a_{11}$, $\vec{b}$ the $p-1$ vector $\vec{b}=(a_{21},\dots,a_{p1})^T=(a_{12},\dots,a_{1p})^T$, and finally, Δ_{p-1} is the following $(p-1)\times(p-1)$ matrix $\Delta_{p-1}=A_{p-1}-\frac{1}{4a}\vec{b}\otimes\vec{b}$.

Let us now get back to the case $q=0$ starting from the beginning (e.g., the zeta function given by Eq. (10), with $q=0$). From that expression, the recurrence (21) can be obtained directly in the same way —this is rather obvious, since q plays no role in the derivation. What is not so obvious is to realize that the limit as $q\to 0$ of Eq. (11) is *exactly* the recurrent

formula (21). More precisely, what is obtained in the limit is the reflected formula which one gets after using the well known Epstein zeta function reflection

$$\Gamma(s) Z(s;A) = \frac{\pi^{2s-p/2}}{\sqrt{\det A}} \Gamma(p/2-s) Z(p/2-s;A^{-1}), \tag{22}$$

being $Z(s;A)$ the Epstein zeta function [15]. After some thinking, such result is easy to understand. Summing up, we have thus checked that our formula (11) is valid for *any* $q \geq 0$, since it contains in a hidden way, for $q=0$, the recurrent expression (21).

As announced at the beginning, the formulas derived here can be considered as generalizations (in more than one sense) of the Chowla-Selberg (CS) formula. All share the same properties that are so much appreciated by number-theoretists as pertaining to the CS formula. In a way, these expressions can be viewed as improved reflection formulas for zeta functions; they are in fact much better than those in several aspects. Namely, while a reflection formula connects one region of the complex plane with a complementary region (with some intersection) by analytical continuation, the CS formula and our formulas are valid on the *whole* complex plane, exhibiting the poles of the zeta function and the corresponding residua *explicitly*. Even more important, while a reflection formula is intended to replace the initial expression of the zeta function —a power series whose convergence can be extremely slow— by another power series with the same type of convergence, it turns out that the expressions here considered give the meromorphic extension of the zeta function, on the whole complex s-plane, in terms of an *exponentially decreasing* power series (as was the case with the CS formula, that one being its most precious property).

Actually, exponential convergence strictly holds under the condition that $q \geq 0$. However, the formulas themselves are valid for $q<0$ or even complex. What is not guaranteed for general $q \in \mathbf{C}$ is the exponential convergence of the series, nor its power-like convergence, for that matter. Those analytical continuations in q must be dealt with specifically, case by case. The physical example of a field theory with a chemical potential falls clearly into this class.

2.3. Particular case $p=2$. The above statements apply both for the cases $q>0$ and $q=0$. The last situation is more involved, however. One is led to employ the recurrence relation (21) several times ($p-1$, in general), what gives rise each time to an additional series of Bessel functions K_ν (exponential convergence). For completeness, let us write down the corresponding series when $p=2$ explicitly. They are, with $q>0$ [13]

$$\zeta_E(s;a,b,c;q) = -q^{-s} + \frac{2\pi q^{1-s}}{(s-1)\sqrt{\Delta}} + \frac{4}{\Gamma(s)} \left[\left(\frac{q}{a}\right)^{1/4} \left(\frac{\pi}{\sqrt{qa}}\right)^s \sum_{n=1}^\infty n^{s-1/2} K_{s-1/2}\left(2\pi n \sqrt{\frac{q}{a}}\right)\right.$$

$$+\sqrt{\frac{q}{a}}\left(2\pi\sqrt{\frac{a}{q\Delta}}\right)^s \sum_{n=1}^\infty n^{s-1} K_{s-1}\left(4\pi n\sqrt{\frac{aq}{\Delta}}\right) \tag{23}$$

$$+\sqrt{\frac{2}{a}}(2\pi)^s \sum_{n=1}^\infty n^{s-1/2}\cos(\pi n b/a)\sum_{d|n} d^{1-2s}\left(\Delta+\frac{4aq}{d^2}\right)^{1/4-s/2} K_{s-1/2}\left(\frac{\pi n}{a}\sqrt{\Delta+\frac{4aq}{d^2}}\right)\Bigg],$$

where $\Delta = 4ac - b^2 > 0$, and, with $q = 0$, the CS formula [10]

$$\zeta_E(s;a,b,c;0) = 2\zeta(2s)\, a^{-s} + \frac{2^{2s}\sqrt{\pi}\, a^{s-1}}{\Gamma(s)\Delta^{s-1/2}}\Gamma(s-1/2)\zeta(2s-1)$$
$$+\frac{2^{s+5/2}\pi^s}{\Gamma(s)\,\Delta^{s/2-1/4}\sqrt{a}}\sum_{n=1}^\infty n^{s-1/2}\sigma_{1-2s}(n)\,\cos(\pi n b/a)\, K_{s-1/2}\left(\frac{\pi n}{a}\sqrt{\Delta}\right). \tag{24}$$

where $\sigma_s(n) \equiv \sum_{d|n} d^s$, sum over the s-powers of the divisors of n. (There is a missprint in the transcription of formula (24) in Ref. [12]). We observe that the rhs's of (23) and (24) exhibit a simple pole at $s = 1$, with common residue:

$$\text{Res}_{s=1}\zeta_E(s;a,b,c;q) = \frac{2\pi}{\sqrt{\Delta}} = \text{Res}_{s=1}\zeta_E(s;a,b,c;0). \tag{25}$$

3. The case of a truncated range

The most involved case in the family of Epstein-like zeta functions corresponds to having to deal with a *truncated* range. This comes about when one imposes boundary conditions of the usual Dirichlet or Neumann type [13]. Jacobi's theta function identity and Poisson's summation formula are then *useless* and no expression in terms of a convergent series for the analytical continuation to values of $\text{Re}\, s$ below the abscissa of convergence can be obtained. The best one gets is an *asymptotic* series expression. However, the issue of extending the CS formula or, better still, the most general expression we have obtained before for inhomogeneous Epstein zeta functions, is not an easy one. This problem has seldom (if ever) been properly addressed in the literature.

3.1. Example 1. To illustrate the issue, let us consider the following simple example in one dimension:

$$\zeta_G(s;a,c;q) \equiv \sum_{n=-\infty}^\infty \Big[a(n+c)^2+q\Big]^{-s}, \quad \text{Re}\, s > 1/2. \tag{26}$$

Associated with this zeta functions, but considerably more difficult to treat, is the truncated series, with indices running from 0 to ∞

$$\zeta_{G_t}(s;a,c;q) \equiv \sum_{n=0}^\infty \Big[a(n+c)^2+q\Big]^{-s}, \quad \text{Re}\, s > 1/2. \tag{27}$$

In this case the Jacobi identity is of no use. How to proceed then? The only way is to employ specific techniques of analytic continuation of zeta functions [13]. The usual method

involves three steps [16]. The first step is elementary: to write the initial series as a Mellin transformed one

$$\sum_{n=0}^{\infty} \left[a(n+c)^2+q\right]^{-s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \int_0^{\infty} dt\, t^{s-1} \exp\left\{-[a(n+c)^2+q]t\right\}. \tag{28}$$

The second is to expand in power series part of the exponential, while leaving always a converging exponential factor,

$$\sum_{n=0}^{\infty} \left[a(n+c)^2+q\right]^{-s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \int_0^{\infty} dt \sum_{m=0}^{\infty} \frac{(-a)^m}{m!}(n+c)^{2m} t^{s+m-1} e^{-qt}. \tag{29}$$

The third and most difficult step is to interchange the order of the two summations —with the aim to obtain a series of zeta functions— what means transforming the second series into an integral along a path on the complex plane, that has to be closed into a circuit (the sum over poles inside reproduces the original series), with a part of it being sent to infinity. Usually, after interchanging the first series and the integral, there is a contribution of this part of the circuit at infinity, what provides in the end an *additional* contribution to the trivial commutation. More important, what one obtains in general through this process is *not* a convergent series of zeta functions, but an asymptotic series [13]. That is, in our example,

$$\sum_{n=0}^{\infty} \left[a(n+c)^2+q\right]^{-s} \sim \sum_{m=0}^{\infty} \frac{(-a)^m \Gamma(m+s)}{m!\,\Gamma(s)\, q^{m+s}} \zeta_H(-2m,c) + \text{ additional terms.} \tag{30}$$

Being more precise, as outcome of the whole process we obtain the following result for the analytic continuation of the zeta function [17]:

$$\begin{aligned} \zeta_{G_t}(s;a,c;q) &\sim \left(\frac{1}{2}-c\right) q^{-s} + \frac{q^{-s}}{\Gamma(s)} \sum_{m=1}^{\infty} \frac{(-1)^m \Gamma(m+s)}{m!} \left(\frac{q}{a}\right)^{-m} \zeta_H(-2m,c) \\ &+ \sqrt{\frac{\pi}{a}}\, \frac{\Gamma(s-1/2)}{2\Gamma(s)} q^{1/2-s} + \frac{2\pi^s}{\Gamma(s)} a^{-1/4-s/2} q^{1/4-s/2} \sum_{n=1}^{\infty} n^{s-1/2} \cos(2\pi n c) K_{s-1/2}(2\pi n \sqrt{q/a}). \end{aligned} \tag{31}$$

(Note that this expression reduces to Eq. (20) in the limit $c \to 0$.) The first series on the rhs is asymptotic [16, 18]. Observe, on the other hand, the singularity structure of this zeta function. Apart from the pole at $s = 1/2$, there is a whole sequence of poles at the negative real axis, for $s = -1/2, -3/2, -5/2, \ldots$, with residua:

$$\text{Res}_{s=1/2-j}\zeta_{G_t}(s;a,c;q) = \frac{(2j-1)!!\, q^j}{j!\, 2^j \sqrt{a}}, \quad j = 0, 1, 2, \ldots \tag{32}$$

3.2. Example 2. As a second example, in order to obtain the analytic continuation to Re $s \leq 1$ of the truncated inhomogeneous Epstein zeta function in two dimensions,

$$\zeta_{E_t}(s;a,b,c;q) \equiv \sum_{m,n=0}^{\infty} (am^2+bmn+cn^2+q)^{-s}, \tag{33}$$

we can proceed in two ways: either by direct calculation following the three steps as explained above or else by using the final formula for the Epstein zeta function in one dimension (example 1) recurrently. In both cases the end result is the same:

$$\begin{aligned}
\zeta_{E_t}(s;a,b,c;q) &\equiv \sum_{m,n=0}^{\infty} (am^2+bmn+cn^2+q)^{-s} \\
&\sim \frac{(4a)^s}{\Gamma(s)} \sum_{m,n=1}^{\infty} \frac{(-1)^m \Gamma(m+s)}{m!} (2a)^{2m} (\Delta n^2 + 4aq)^{-m-s} \zeta_H\left(-2m; \frac{bn}{2a}\right) \\
&\quad - \frac{b\,q^{1-s}}{(s-1)\Delta\Gamma(s-1)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+s-1) B_n}{n!} \left(\frac{4aq}{\Delta}\right)^{-n} + \frac{q^{-s}}{4} + \frac{\pi q^{1-s}}{2(s-1)\sqrt{\Delta}} \\
&\quad + \frac{1}{4}\left(\sqrt{\frac{\pi}{a}} + \sqrt{\frac{\pi}{c}}\right) \frac{\Gamma(s-1/2)}{\Gamma(s)} q^{1/2-s} + \frac{1}{\Gamma(s)} \left[2 \left(\frac{q}{a}\right)^{1/4} \left(\frac{\pi}{\sqrt{qa}}\right)^s \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}\left(2\pi n \sqrt{\frac{q}{a}}\right) \right. \\
&\quad + \left(\frac{aq}{\Delta}\right)^{1/4} \left(\pi \sqrt{\frac{a}{q\Delta}}\right)^s \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}\left(2\pi n \sqrt{\frac{aq}{\Delta}}\right) + \sqrt{\frac{q}{a}} \left(2\pi \sqrt{\frac{a}{q\Delta}}\right)^s \sum_{n=1}^{\infty} n^{s-1} K_{s-1}\left(4\pi n \sqrt{\frac{aq}{\Delta}}\right) \\
&\quad \left. + \sqrt{\frac{2}{a}} (2\pi)^s \sum_{n=1}^{\infty} n^{s-1/2} \cos(\pi n b/a) \sum_{d|n} d^{1-2s} \left(\Delta + \frac{4aq}{d^2}\right)^{1/4-s/2} K_{s-1/2}\left(\frac{\pi n}{a} \sqrt{\Delta + \frac{4aq}{d^2}}\right) \right].
\end{aligned} \tag{34}$$

The first series on the rhs is in general asymptotic, although it converges for a wide range of values of the parameters. The second series is always asymptotic and its first term contributes to the pole at $s=1$. As in the case of Eq. (23), the pole structure is here explicit, although much more elaborate. Apart from the pole at $s=1$, whose residue is

$$\text{Res}_{s=1} \zeta_{E_t}(s;a,b,c;q) = \frac{\pi}{2\sqrt{\Delta}} - \frac{b}{\Delta}, \tag{35}$$

there is here also a sequence of poles at $s = \pm 1/2, -3/2, -5/2, \ldots$, with residua:

$$\text{Res}_{s=1/2-j} \zeta_{E_t}(s;a,b,c;q) = \frac{(2j-1)!!\, q^j}{j!\, 2^{j+2}} \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}\right), \quad j = 0,1,2,\ldots \tag{36}$$

The formula above, Eq. (34), is really imposing and hints already towards the conclusion that the derivation of a general expression in p dimensions for the zeta function considered in Sect. 2 but with a truncated range is not an easy task.

4. Some uses of the formulas

These formulas are very powerful expressions in order to determine the analytic structure of generalized inhomogeneous Epstein type zeta functions, to obtain specific values of these

zeta functions at different points, and from there, in particular, the Casimir effect and heat kernel coefficients, and also in order to calculate derivatives of the zeta function, and from them, in particular, the associated determinant. Notice that obtaining derivatives of the formulas in Sect. 2 presents no problem. Only for truncated zeta functions (Sect. 3) the usual care must be taken when dealing with asymptotical expansions. We shall illustrate these uses with three specific applications.

4.1. Application 1. In a recent paper by R. Bousso and S. Hawking [19], where the *trace anomaly* of a dilaton coupled scalar in two dimensions is calculated, the zeta function method is employed for obtaining the one-loop effective action, W, which is given by the well known expression

$$W = \frac{1}{2}\left[\zeta_A(0)\ln\mu^2 + {\zeta_A}'(0)\right], \tag{37}$$

with $\zeta_A(s) = \mathrm{tr}\ A^{-s}$. In conformal field theory and in a Euclidean background manifold of toroidal topology, the eigenvalues of A are found perturbatively (see [19]), what leads one to consider the following zeta function:

$$\zeta_A(s) = \sum_{k,l=-\infty}^{\infty} (\Lambda_{kl})^{-s}, \tag{38}$$

with the eigenvalues Λ_{kl} being given by

$$\Lambda_{kl} = k^2 + l^2 + \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2(4l^2-1)}, \tag{39}$$

where ϵ is a perturbation parameter. It can be shown that the integral of the trace anomaly is given by the value of the zeta function at $s = 0$. One barely needs to follow the several pages long discussion in [19], leading to the calculation of this value, in order to appreciate the power of the formulae of the preceding section. In fact, to begin with, no mass term needs to be introduced to arrive at the result and no limit mass $\to 0$ needs to be taken later. Using binomial expansion (the same as in Ref. [19]), one gets

$$\zeta(s) = \sum_{k,l=-\infty}^{\infty}\left(k^2 + l^2 + \frac{\epsilon^2}{2}\right)^{-s} - \frac{\epsilon^2 s}{2}\sum_{k,l=-\infty}^{\infty}\left(k^2 + l^2 + \frac{\epsilon^2}{2}\right)^{-1-s}(4l^2-1)^{-1}. \tag{40}$$

From Eq. (23) above, the first zeta function gives, at $s = 0$, *exactly*: $-\pi\epsilon^2/2$. And this is the whole result (which does coincide with the one obtained in [19]), since the second term has no pole at $s = 0$ and provides no contribution.

4.2. Application 2. Another direct application is the calculation of the *Casimir energy density* corresponding to a massive scalar field on a general, p dimensional toroidal manifold (see

[20]). In the spacetime $\mathcal{M} = \mathbf{R} \times\Sigma$, with $\Sigma = [0,1]^p/\!\sim$, which is topologically equivalent to the p torus, the Casimir energy density for a massive scalar field is given directly by Eq. (11) at $s=-1/2$, with $q=m^2$ (mass of the field), $\vec{b}=\vec{0}$, and A being the matrix of the metric g on Σ, the general p-torus:

$$E^C_{\mathcal{M},m} = \zeta_{g,\vec{0},m^2}(s=-1/2). \tag{41}$$

The components of g are, in fact, the coefficients of the different terms of the Laplacian, which is the relevant operator in the Klein-Gordon field equation. The massless case is also obtained, with the same specifications, from the corresponding formula Eq. (21). In both cases no extra calculation needs to be done, and the physical results follows from a mere *identification* of the components of the matrix A with those of the metric tensor of the manifold in question [20]. Very much related with this application but more involved and ambitious is the calculation of vacuum energy densities corresponding to spherical configurations and the bag model (see [21, 22], and the many references therein).

4.3. Application 3. A third application consists in calculating the *determinant* of a differential operator, say the Laplacian on a general p-dimensional torus. A very important problem related with this issue is that of the associated anomaly (called the multiplicative or noncommutative anomaly) [23]. To this end the derivative of the zeta function at $s=0$ has to be obtained. From Eq. (11), we get

$$\begin{aligned}
{\zeta'}_{A,\vec{c},q}(0) &= \frac{4(2q)^{p/4}}{\sqrt{\det A}} \sum_{\vec{m}\in\mathbf{Z}^p_{1/2}}{}' \frac{\cos(2\pi\vec{m}\cdot\vec{c})}{(\vec{m}^TA^{-1}\vec{m})^{p/4}}\, K_{p/2}\left(2\pi\sqrt{2q\,\vec{m}^TA^{-1}\vec{m}}\right) \\
&\quad + \begin{cases} \dfrac{(2\pi)^{p/2}\Gamma(-p/2)q^{p/2}}{\sqrt{\det A}}, & p \text{ odd}, \\[2mm] \dfrac{(-1)^k(2\pi)^kq^k}{k!\,\sqrt{\det A}}\,[\Psi(k+1)+\gamma-\ln q]\,, & p=2k \text{ even}, \end{cases}
\end{aligned} \tag{42}$$

and, from here, $\det A = \exp -\zeta'_A(0)$. For $p=2$, we have explicitly:

$$\begin{aligned}
\det A(a,b,c;q) &= e^{2\pi(q-\ln q)/\sqrt{\Delta}}\left(1-e^{-2\pi\sqrt{q/a}}\right)\exp\left\{-4\sum_{n=1}^\infty \frac{1}{n}\left[\sqrt{\frac{a}{q}}\,K_1\left(4\pi n\sqrt{\frac{aq}{\Delta}}\right)\right.\right. \\
&\quad \left.\left. +\cos(\pi nb/a)\sum_{d|n} d\,\exp\left(-\frac{\pi n}{a}\sqrt{\Delta+\frac{4aq}{d^2}}\right)\right]\right\}.
\end{aligned} \tag{43}$$

In the homogeneous case (CS formula) we obtain for the determinant:

$$\det A(a,b,c) = \frac{1}{a}\exp\left[-4\zeta'(0) - \frac{\pi\sqrt{\Delta}}{6a} - 4\sum_{n=1}^\infty \frac{\sigma_1(n)}{n}\cos(\pi nb/a)e^{-\pi n\sqrt{\Delta}/a}\right], \tag{44}$$

or, in terms of the Teichmüller coefficients, τ_1 and τ_2, of the metric tensor (for the metric, A, corresponding to the general torus in two dimensions):

$$\det A(\tau_1, \tau_2) = \frac{\tau_2}{4\pi^2|\tau|^2} \exp \left[-4\zeta'(0) - \frac{\pi\tau_2}{3|\tau|^2} - 4 \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} \cos \left(\frac{2\pi n \tau_1}{|\tau|^2} \right) e^{-\pi n \tau_2/|\tau|^2} \right]. \quad (45)$$

Needless to mention, all the good properties of the expression for the zeta function are just transferred to the associated determinants, which are thus given, on its turn, in terms of very quickly convergent series.

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